

Limit cycle in a multidimensional system

The homotopy analysis method is successfully applied by Liao [39] to solve limit cycles of one-dimensional nonlinear dynamical systems, governed by

$$\ddot{u}(t) = f(u, \dot{u}, \ddot{u}), \quad (13.1)$$

where t denotes the time, the dot denotes derivative with respect to t , and $f(u, \dot{u}, \ddot{u})$ is a known function of u, \dot{u} , and \ddot{u} . Unlike perturbation techniques, it is unnecessary to assume the existence of any small/large quantities in the above equation. Here, we show that the homotopy analysis method can also be applied to gain limit cycles of multidimensional dynamical systems.

As an example, let us consider a two-dimensional nonlinear dynamical system governed by (see Kahn [103])

$$\ddot{x} + x = \epsilon \dot{x}(1 - x^2 w), \quad (13.2)$$

$$\dot{w} = -\epsilon (w^2 - \mu x^4), \quad (13.3)$$

where the dot denotes differentiation with respect to t , μ and ϵ are physical parameters, x and w are two unknown functions. Physically, a limit cycle is independent of initial conditions. Let T and $\alpha = \max[x(t)]$ denote the period and the maximum value of $x(t)$ of the limit cycle, respectively. Without loss of generality, we can define $t = 0$ so that

$$x(0) = \alpha, \quad \dot{x}(0) = 0. \quad (13.4)$$

Define

$$\delta = \frac{1}{T} \int_0^T w(t) dt \quad (13.5)$$

and let

$$\omega = T/2\pi$$

denote the frequency of $x(t)$ of the limit cycle. Under the transformations

$$\tau = \omega t, \quad x(t) = \alpha u(\tau), \quad w(t) = \delta + v(\tau), \quad (13.6)$$

Equations (13.2) and (13.3) become

$$\omega^2 u'' + u = \epsilon \omega u' (1 - \alpha^2 \delta u^2 - \alpha^2 u^2 v), \quad (13.7)$$

$$\omega v' = -\epsilon (\delta^2 + 2\delta v + v^2 - \mu \alpha^4 u^4), \quad (13.8)$$

subject to the initial conditions

$$u(0) = 1, \quad u'(0) = 0, \quad (13.9)$$

where the prime represents differentiation with respect to τ . Furthermore, from (13.5) and (13.6), it holds that

$$\int_0^{2\pi} v(\tau) d\tau = 0, \quad (13.10)$$

which provides us with the condition for $v(\tau)$. Note that α, δ , and ω are unknown.

13.1 Homotopy analysis solution

13.1.1 Zero-order deformation equation

From a physical point of view, a limit cycle can be expressed by periodic functions. Clearly, $u(\tau)$ and $v(\tau)$ may be expressed in the forms:

$$u(\tau) = \sum_{n=1}^{+\infty} [a_n \cos(n\tau) + b_n \sin(n\tau)] \quad (13.11)$$

and

$$v(\tau) = \sum_{n=1}^{+\infty} [c_n \cos(n\tau) + d_n \sin(n\tau)], \quad (13.12)$$

where a_n, b_n, c_n , and d_n are coefficients. The above expressions provide the so-called *rules of solution expression* for $u(\tau)$ and $v(\tau)$, respectively.

Under the above *rules of solution expressions* and from the initial conditions (13.9) and (13.10), it is convenient to choose

$$u_0(\tau) = \cos \tau, \quad v_0(\tau) = 0 \quad (13.13)$$

as the initial guesses of $u(\tau)$ and $v(\tau)$. Here, $v_0(\tau) = 0$ is chosen because of the lack of information about $v(\tau)$, especially the relationship between $u(\tau)$ and $v(\tau)$. Let α_0, δ_0 , and ω_0 denote the initial guesses of α, δ , and ω , respectively. Under the *rules of solution expression* denoted by (13.11) and (13.12) and from Equations (13.7) and (13.8), we choose the auxiliary linear operators

$$\mathcal{L}_u f = \frac{\partial^2 f}{\partial \tau^2} + f \quad (13.14)$$

and

$$\mathcal{L}_v f = \frac{\partial f}{\partial \tau} \quad (13.15)$$

with the properties

$$\mathcal{L}_u (C_1 \cos \tau + C_2 \sin \tau) = 0, \quad \mathcal{L}_v (C_3) = 0, \quad (13.16)$$

respectively, where C_1, C_2 , and C_3 are coefficients and f is a real function. For simplicity, we define from Equations (13.7) and (13.8) the nonlinear operators

$$\begin{aligned} & \mathcal{N}_u [U(\tau; q), V(\tau; q), A(q), \Delta(q), \Omega(q)] \\ &= \Omega^2(q) \frac{\partial^2 U(\tau; q)}{\partial \tau^2} + U(\tau; q) \\ &- \epsilon \Omega(q) \frac{\partial U(\tau; q)}{\partial \tau} [1 - A^2(q) \Delta(q) U^2(\tau; q) - A^2(q) U^2(\tau; q) V(\tau; q)] \end{aligned} \quad (13.17)$$

and

$$\begin{aligned} & \mathcal{N}_v [U(\tau; q), V(\tau; q), A(q), \Delta(q), \Omega(q)] \\ &= \Omega(q) \frac{\partial V(\tau; q)}{\partial \tau} \\ &+ \epsilon [\Delta^2(q) + 2\Delta(q) V(\tau; q) + V^2(\tau; q) - \mu A^4(q) U^4(\tau; q)], \end{aligned} \quad (13.18)$$

where $q \in [0, 1]$ is the imbedding parameter, $U(\tau; q)$ and $V(\tau; q)$ are real functions of τ and q , $A(q)$, $\Delta(q)$, and $\Omega(q)$ are real functions of q .

Let \hbar_u and \hbar_v denote the nonzero auxiliary parameters, $H_u(\tau)$ and $H_v(\tau)$ the nonzero auxiliary functions, respectively. We construct the zero-order deformation equations

$$\begin{aligned} & (1 - q) \mathcal{L}_u [U(\tau; q) - u_0(\tau)] \\ &= q \hbar_u H_u(\tau) \mathcal{N}_u [U(\tau; q), V(\tau; q), A(q), \Delta(q), \Omega(q)], \end{aligned} \quad (13.19)$$

$$\begin{aligned} & (1 - q) \mathcal{L}_v [V(\tau; q) - v_0(\tau)] \\ &= q \hbar_v H_v(\tau) \mathcal{N}_v [U(\tau; q), V(\tau; q), A(q), \Delta(q), \Omega(q)], \end{aligned} \quad (13.20)$$

subject to the conditions

$$U(0; q) = 1, \quad \left. \frac{\partial U(\tau; q)}{\partial \tau} \right|_{\tau=0} = 0, \quad \int_0^{2\pi} V(\tau; q) d\tau = 0, \quad (13.21)$$

where $\tau \geq 0$ and $q \in [0, 1]$.

When $q = 0$, it is clear from (13.13) and the above zero-order deformation equations that

$$U(\tau; 0) = u_0(\tau), \quad V(\tau; 0) = v_0(\tau). \quad (13.22)$$

When $q = 1$, Equations (13.19) to (13.21) are equivalent to Equations (13.7) to (13.10), respectively, provided

$$U(\tau; 1) = u(\tau), \quad V(\tau; 1) = v(\tau) \quad (13.23)$$

and

$$A(1) = \alpha, \quad \Delta(1) = \delta, \quad \Omega(1) = \omega. \quad (13.24)$$

So, as the embedding parameter q increases from 0 to 1, $U(\tau; q)$, and $V(\tau; q)$ vary from the initial guesses $u_0(\tau)$ and $v_0(\tau)$ to the exact solutions $u(\tau)$ and $v(\tau)$, respectively, so do $A(q)$, $\Delta(q)$, and $\Omega(q)$ from the initial guesses α_0 , δ_0 , and ω_0 to the corresponding exact values α , δ , and ω .

The zero-order deformation equations (13.19) and (13.20) contain the two auxiliary parameters \hbar_u, \hbar_v and the two auxiliary functions $H_u(\tau), H_v(\tau)$. Assume that all of them are properly chosen so that the terms

$$u_n(\tau) = \left(\frac{1}{n!} \right) \frac{d^n U(\tau; q)}{dq^n} \Big|_{q=0}, \quad (13.25)$$

$$v_n(\tau) = \left(\frac{1}{n!} \right) \frac{d^n V(\tau; q)}{dq^n} \Big|_{q=0}, \quad (13.26)$$

and

$$\alpha_n = \left(\frac{1}{n!} \right) \frac{d^n A(q)}{dq^n} \Big|_{q=0}, \quad (13.27)$$

$$\delta_n = \left(\frac{1}{n!} \right) \frac{d^n \Delta(q)}{dq^n} \Big|_{q=0}, \quad (13.28)$$

$$\omega_n = \left(\frac{1}{n!} \right) \frac{d^n \Omega(q)}{dq^n} \Big|_{q=0} \quad (13.29)$$

exist for $n \geq 1$. Then, using Taylor's theorem and (13.22), we have the power series of q in the forms:

$$U(\tau; q) = u_0(\tau) + \sum_{n=1}^{+\infty} u_n(\tau) q^n, \quad (13.30)$$

$$V(\tau; q) = v_0(\tau) + \sum_{n=1}^{+\infty} v_n(\tau) q^n, \quad (13.31)$$

$$A(q) = \alpha_0 + \sum_{n=1}^{+\infty} \alpha_n q^n, \quad (13.32)$$

$$\Delta(q) = \delta_0 + \sum_{n=1}^{+\infty} \delta_n q^n, \quad (13.33)$$

$$\Omega(q) = \omega_0 + \sum_{n=1}^{+\infty} \omega_n q^n. \quad (13.34)$$

Assuming that \hbar_u , \hbar_v , $H_u(\tau)$, and $H_v(\tau)$ are properly chosen so that the above series are convergent at $q = 1$, we obtain, using (13.23) and (13.24),

the solution series

$$u(\tau) = u_0(\tau) + \sum_{n=1}^{+\infty} u_n(\tau), \quad (13.35)$$

$$v(\tau) = v_0(\tau) + \sum_{n=1}^{+\infty} v_n(\tau), \quad (13.36)$$

$$\alpha = \alpha_0 + \sum_{n=1}^{+\infty} \alpha_n, \quad (13.37)$$

$$\delta = \delta_0 + \sum_{n=1}^{+\infty} \delta_n, \quad (13.38)$$

and

$$\omega = \omega_0 + \sum_{n=1}^{+\infty} \omega_n. \quad (13.39)$$

13.1.2 High-order deformation equation

For conciseness, define the vectors

$$\vec{u}_k = \{u_0(\tau), u_1(\tau), \dots, u_k(\tau)\}, \quad \vec{v}_k = \{v_0(\tau), v_1(\tau), \dots, v_k(\tau)\}, \quad (13.40)$$

$$\vec{\alpha}_k = \{\alpha_0, \alpha_1, \dots, \alpha_k\}, \quad \vec{\delta}_k = \{\delta_0, \delta_1, \dots, \delta_k\}, \quad (13.41)$$

and

$$\vec{\omega}_k = \{\omega_0, \omega_1, \dots, \omega_k\}. \quad (13.42)$$

Differentiating the zero-order deformation equations (13.19) to (13.21) n times with respect to q , then dividing by $n!$, and finally setting $q = 0$, we have the high-order deformation equations

$$\begin{aligned} & \mathcal{L}_u [u_n(\tau) - \chi_n u_{n-1}(\tau)] \\ &= \hbar_u H_u(\tau) R_n^u(\vec{u}_{n-1}, \vec{v}_{n-1}, \vec{\alpha}_{n-1}, \vec{\delta}_{n-1}, \vec{\omega}_{n-1}), \end{aligned} \quad (13.43)$$

$$\begin{aligned} & \mathcal{L}_v [v_n(\tau) - \chi_n v_{n-1}(\tau)] \\ &= \hbar_v H_v(\tau) R_n^v(\vec{u}_{n-1}, \vec{v}_{n-1}, \vec{\alpha}_{n-1}, \vec{\delta}_{n-1}, \vec{\omega}_{n-1}), \end{aligned} \quad (13.44)$$

subject to the conditions

$$u_n(0) = 0, \quad u'_n(0) = 0, \quad \int_0^{2\pi} v_n(\tau) d\tau = 0, \quad (13.45)$$

where χ_n is defined by (2.42),

$$R_n^u(\vec{u}_{n-1}, \vec{v}_{n-1}, \vec{\alpha}_{n-1}, \vec{\delta}_{n-1}, \vec{\omega}_{n-1})$$

$$\begin{aligned}
&= \frac{1}{(n-1)!} \frac{d^{n-1} \mathcal{N}_u [U(\tau; q), V(\tau; q), A(q), \Delta(q), \Omega(q)]}{d q^{n-1}} \\
&= \sum_{j=0}^{n-1} u''_{n-1-j}(\tau) \left(\sum_{i=0}^j \omega_i \omega_{j-i} \right) + u_{n-1}(\tau) - \epsilon F_{n-1}(\tau) \\
&+ \epsilon \sum_{j=0}^{n-1} F_{n-1-j}(\tau) \sum_{i=0}^j [\delta_i + v_i(\tau)] W_{j-i}(\tau)
\end{aligned} \tag{13.46}$$

and

$$\begin{aligned}
&R_n^v(\vec{u}_{n-1}, \vec{v}_{n-1}, \vec{\alpha}_{n-1}, \vec{\delta}_{n-1}, \vec{\omega}_{n-1}) \\
&= \frac{1}{(n-1)!} \frac{d^{n-1} \mathcal{N}_v [U(\tau; q), V(\tau; q), A(q), \Delta(q), \Omega(q)]}{d q^{n-1}} \\
&= \sum_{j=0}^{n-1} \omega_j v'_{n-1-j}(\tau) + \epsilon \sum_{j=0}^{n-1} [\delta_j \delta_{n-1-j} + 2\delta_j v_{n-1-j}(\tau)] \\
&+ \epsilon \sum_{j=0}^{n-1} [v_j(\tau) v_{n-1-j}(\tau) - \mu W_j(\tau) W_{n-1-j}(\tau)],
\end{aligned} \tag{13.47}$$

under the definitions

$$F_k(\tau) = \sum_{j=0}^k \omega_{k-j} u'_j(\tau), \tag{13.48}$$

$$W_k(\tau) = \sum_{j=0}^k \left(\sum_{m=0}^{k-j} \alpha_m \alpha_{k-j-m} \right) \left[\sum_{n=0}^j u_n(\tau) u_{j-n}(\tau) \right]. \tag{13.49}$$

It should be emphasized that the linear high-order deformation equations (13.43) and (13.44) are uncoupled and can be easily solved.

There are five unknowns: $u_n(\tau)$, $v_n(\tau)$, α_{n-1} , δ_{n-1} , and ω_{n-1} , and we have only Equations (13.43), (13.44), and (13.45) for $u_n(\tau)$ and $v_n(\tau)$. Therefore, the problem is not closed and three additional algebraic equations are needed to determine α_{n-1} , δ_{n-1} , and ω_{n-1} . Under the *rules of solution expression* denoted by (13.11) and (13.12) and from Equations (13.43) and (13.44), $H_u(\tau)$ and $H_v(\tau)$ may be sine and cosine functions. For simplicity, we select

$$H_u(\tau) = H_v(\tau) = 1. \tag{13.50}$$

When $n = 1$, by substituting (13.13) into (13.46) and (13.47), we gain

$$R_1^u = a_{1,0} \cos \tau + b_{1,0} \sin \tau + b_{1,1} \sin(3\tau) \tag{13.51}$$

and

$$R_1^v = c_{1,0} + c_{1,1} \cos(2\tau) + c_{1,2} \cos(4\tau), \tag{13.52}$$

where $a_{1,0}$, $b_{1,0}$, $b_{1,1}$, $c_{1,0}$, $c_{1,1}$, and $c_{1,2}$ are coefficients independent of τ . If $a_{1,0} \neq 0$ and $b_{1,0} \neq 0$, from the property (13.16) of \mathcal{L}_u , the solution $u_1(\tau)$ of Equation (13.43) contains the so-called secular terms $\tau \sin \tau$ and $\tau \cos \tau$, which do not conform to the *rule of solution expression* denoted by (13.11). Moreover, if $c_{1,0} \neq 0$, from the property (13.16) of \mathcal{L}_v , the solution $v_1(\tau)$ of Equation (13.44) contains the secular term $c_{1,0} \tau$, which disregards the *rule of solution expression* denoted by (13.12). In order to conform to the *rules of solution expression* denoted by (13.43) and (13.44), we must enforce

$$a_{1,0} = 0, \quad b_{1,0} = 0, \quad c_{1,0} = 0,$$

which provides us with three additional algebraic equations

$$\omega_0 - \frac{\alpha_0^2 \delta_0}{4} = 0, \quad \omega_0^2 - 1 = 0, \quad \delta_0^2 - \frac{3\alpha_0^4 \mu}{8} = 0, \quad (13.53)$$

whose solutions are

$$\alpha_0 = \frac{2}{\sqrt[8]{6\mu}}, \quad \delta_0 = \sqrt[4]{6\mu}, \quad \omega_0 = 1. \quad (13.54)$$

Now the problem is solved in accordance with the *rules of solution expression* denoted by (13.11) and (13.12). We now have

$$R_1^u = b_{1,1} \sin(3\tau)$$

and

$$R_1^v = c_{1,1} \cos(2\tau) + c_{1,2} \cos(4\tau).$$

Solving the first-order deformation equations (13.43) and (13.44) under the conditions noted in (13.45), we have

$$u_1(\tau) = -\left(\frac{\epsilon}{8}\right) \hbar_u (3 \sin \tau - \sin 3\tau) \quad (13.55)$$

and

$$v_1(\tau) = -\left(4\epsilon \sqrt{\frac{\mu}{6}}\right) \hbar_v \left(\sin 2\tau + \frac{1}{8} \sin 4\tau\right). \quad (13.56)$$

Similarly, first solving a set of linear algebraic equations

$$\epsilon^2 \left(3\hbar_u + \sqrt[4]{176\mu} \hbar_v\right) - 48 \omega_1 = 0, \quad (13.57)$$

$$(246\mu^3)^{1/8} \alpha_1 + (46)^{3/4} \delta_1 = 0, \quad (13.58)$$

$$(126\mu)^{3/8} \alpha_1 - \delta_1 = 0 \quad (13.59)$$

to gain α_1 , δ_1 , and ω_1 , and then the remaining second-order deformation equations (13.43) to (13.45), we obtain $u_2(\tau)$ and $v_2(\tau)$. In this way we successively obtain α_{n-1} , δ_{n-1} , ω_{n-1} , $u_n(\tau)$, and $v_n(\tau)$.

At the n th-order of approximation, $u(\tau)$ and $v(\tau)$ can be expressed by

$$u(\tau) = \sum_{k=0}^{M_n^u} [a_{n,k} \cos(2k+1)\tau + b_{n,k} \sin(2k+1)\tau]$$

and

$$v(\tau) = \sum_{k=1}^{M_n^v} [c_{n,k} \cos(2k\tau) + d_{n,k} \sin(2k\tau)],$$

where M_n^u and M_n^v are integers dependent upon the order n of approximation. Thus, the frequency of the motion $w(t)$ is twice that of $x(t)$.

13.1.3 Convergence theorem

THEOREM 13.1

If the solution series (13.35) to (13.39) are convergent, where $u_n(\tau)$ and $v_n(\tau)$ are governed by Equations (13.43) to (13.45) under the definitions (13.46) to (13.49), and (2.42), they must be the solution of Equations (13.7) to (13.10).

Proof: If the solution series (13.35) and (13.36) are convergent, then

$$\lim_{m \rightarrow +\infty} u_m(\tau) = 0, \quad \lim_{m \rightarrow +\infty} v_m(\tau) = 0.$$

From Equation (13.43) and using the definitions (2.42) and (13.14), we then have

$$\begin{aligned} & \hbar_u H_u(\tau) \sum_{n=1}^{+\infty} R_n^u(\vec{u}_{n-1}, \vec{v}_{n-1}, \vec{\alpha}_{n-1}, \vec{\delta}_{n-1}, \vec{\omega}_{n-1}) \\ &= \sum_{n=1}^{+\infty} \mathcal{L}_u [u_n(\tau) - \chi_n u_{n-1}(\tau)] \\ &= \lim_{m \rightarrow +\infty} \sum_{n=1}^m \mathcal{L}_u [u_n(\tau) - \chi_n u_{n-1}(\tau)] \\ &= \lim_{m \rightarrow +\infty} \mathcal{L}_u [u_m(\tau)] \\ &= \mathcal{L}_u \left[\lim_{m \rightarrow +\infty} u_m(\tau) \right] \\ &= 0, \end{aligned}$$

which yields, since $\hbar_u \neq 0$ and $H_u(\tau) \neq 0$,

$$\sum_{n=1}^{+\infty} R_n^u(\vec{u}_{n-1}, \vec{v}_{n-1}, \vec{\alpha}_{n-1}, \vec{\delta}_{n-1}, \vec{\omega}_{n-1}) = 0.$$

Similarly,

$$\sum_{n=1}^{+\infty} R_n^v(\vec{u}_{n-1}, \vec{v}_{n-1}, \vec{\alpha}_{n-1}, \vec{\delta}_{n-1}, \vec{\omega}_{n-1}) = 0.$$

Substituting (13.46) and (13.47) into the above expressions and simplifying them, we have, due to the convergence of the series (13.37) to (13.39), that

$$\begin{aligned} & \left(\sum_{i=0}^{+\infty} \omega_i \right)^2 \frac{d^2}{d\tau^2} \left[\sum_{j=0}^{+\infty} u_j(\tau) \right] + \sum_{j=0}^{+\infty} u_j(\tau) \\ &= \epsilon \left(\sum_{i=0}^{+\infty} \omega_i \right) \frac{d}{d\tau} \left[\sum_{j=0}^{+\infty} u_j(\tau) \right] \\ & \times \left\{ 1 - \left(\sum_{i=0}^{+\infty} \alpha_i \right)^2 \left(\sum_{j=0}^{+\infty} u_j \right)^2 \sum_{k=0}^{+\infty} [\delta_k + v_k(\tau)] \right\} \end{aligned}$$

and

$$\begin{aligned} & \left(\sum_{i=0}^{+\infty} \omega_i \right) \frac{d}{d\tau} \left[\sum_{j=0}^{+\infty} v_j(\tau) \right] \\ &= -\epsilon \left\{ \left[\sum_{k=0}^{+\infty} \delta_k + \sum_{k=0}^{+\infty} v_k(\tau) \right]^2 - \mu \left(\sum_{i=0}^{+\infty} \alpha_i \right)^4 \left(\sum_{j=0}^{+\infty} u_j \right)^4 \right\}. \end{aligned}$$

From (13.13) and (13.45), we have

$$\sum_{i=0}^{+\infty} u_i(0) = 1, \quad \sum_{i=0}^{+\infty} u'_i(0) = 0, \quad \int_0^{2\pi} \left[\sum_{i=0}^{+\infty} v_i(\tau) \right] d\tau = 0.$$

Comparing the above equations with Equations (13.7) to (13.10), it is obvious that the series (13.35) to (13.39) are the solutions. This ends the proof.

13.2 Result analysis

According to Theorem 13.1 we should ensure that the solution series (13.35) to (13.39) converge. Note that these solution series contain two auxiliary parameters \hbar_u and \hbar_v . For simplicity, let

$$\hbar_u = \hbar_v = \hbar$$

so that the approximations of $u(\tau)$, $v(\tau)$, ω , α , and δ are dependent only on \hbar . Generally, for any given physical parameters ϵ and μ , we first investigate the influence of the auxiliary parameter \hbar on the convergence of the series by plotting the so-called \hbar -curves (see page 26 and §3.5.1) of α , δ , and ω . For example, when $\epsilon = 1/5$ and $\mu = 3$ the \hbar -curves are as shown in Figure 13.1, clearly indicating the valid regions of \hbar for the corresponding series of α , δ , and ω . Obviously, when $\epsilon = 1/5$ and $\mu = 3$, the solution series (13.37) to (13.39) converge if $-3/2 < \hbar < 0$. For instance, when $\hbar_u = \hbar_v = -3/4$, the solution series of ω , α , and δ converge to 0.96968, 1.41399, and 2.07015, respectively, as shown in Table 13.1. We can employ the so-called homotopy-Padé technique (see page 38 and §3.5.2) to accelerate the convergence, as shown in Table 13.2. As long as the solution series of α , δ , and ω are convergent, the corresponding series of $u(\tau)$ and $v(\tau)$ given by the same value of \hbar also converge, as shown in Figures 13.2 to 13.4 when $\epsilon = 1/5$ and $\mu = 3$.

In this way, for any given physical parameters ϵ and μ , we can gain convergent analytic results of the limit cycle of the two-dimensional dynamical system. As ϵ increases, the nonlinearity becomes stronger so that a higher order of approximation is necessary. For example, when $\epsilon = 3/4$ and $\mu = 1$, the \hbar -curves of α , δ , and ω clearly indicate that the solution series converge when $\hbar = -3/4$, as shown in Figure 13.5. However, higher-order approximations are needed to get accurate enough results, as shown in Figures 13.6 to 13.8.

The homotopy analysis method was applied to solve limit cycles of one-dimensional systems [39]. This example shows that the homotopy analysis method may be employed to gain limit-cycles of multidimensional systems.

TABLE 13.1

The m th-order approximations of ω, α , and δ when $\epsilon = 1/5, \mu = 3$ by means of $\hbar_u = \hbar_v = -3/4$ and $H_u(\tau) = H_v(\tau) = 1$.

m	ω	α	δ
1	1.00000	1.39354	2.05977
2	0.97063	1.40476	2.06318
3	0.96966	1.41020	2.06458
4	0.96963	1.41251	2.06668
5	0.96968	1.41346	2.06843
6	0.96969	1.41382	2.06944
7	0.96969	1.41395	2.06989
8	0.96968	1.41398	2.07006
9	0.96968	1.41399	2.07011
10	0.96968	1.41399	2.07013
11	0.96968	1.41399	2.07014
12	0.96968	1.41399	2.07015
13	0.96968	1.41399	2.07015
14	0.96968	1.41399	2.07015

TABLE 13.2

The $[m, m]$ homotopy-Padé approximations of ω, α , and δ when $\epsilon = 1/5, \mu = 3$ by means of $\hbar_u = \hbar_v = -3/4$ and $H_u(\tau) = H_v(\tau) = 1$.

$[m, m]$	ω	α	δ
$[1, 1]$	0.96889	1.39354	2.05977
$[2, 2]$	0.96977	1.41413	2.08345
$[3, 3]$	0.96968	1.41414	2.08735
$[4, 4]$	0.96968	1.41399	2.07015
$[5, 5]$	0.96968	1.41398	2.07016
$[6, 6]$	0.96968	1.41399	2.07015
$[7, 7]$	0.96968	1.41399	2.07015

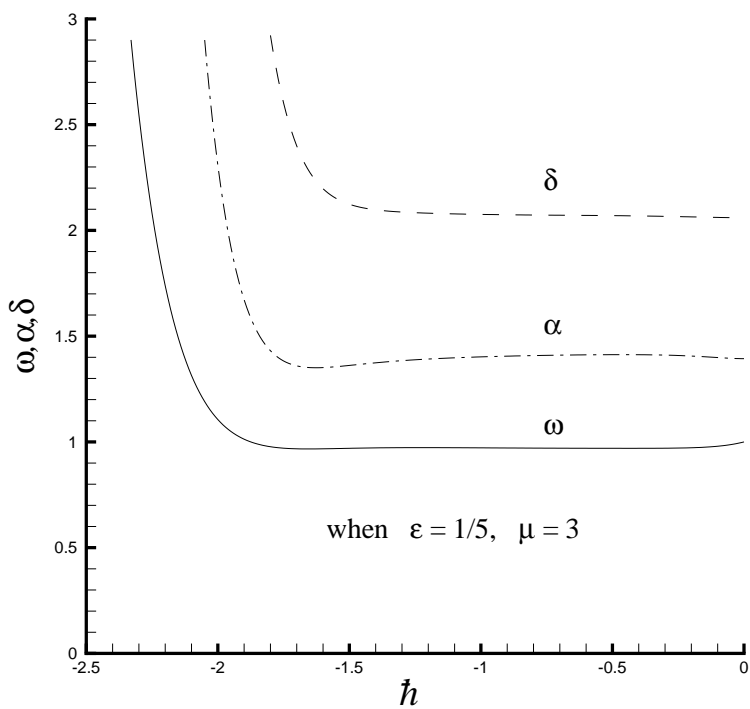


FIGURE 13.1

The \hbar -curves of ω, α , and δ at the 10th order of approximation when $\epsilon = 1/5$ and $\mu = 3$ by means of $H_u(\tau) = H_v(\tau) = 1$. Dashed line: δ ; dash-dotted line: α ; solid line: ω .

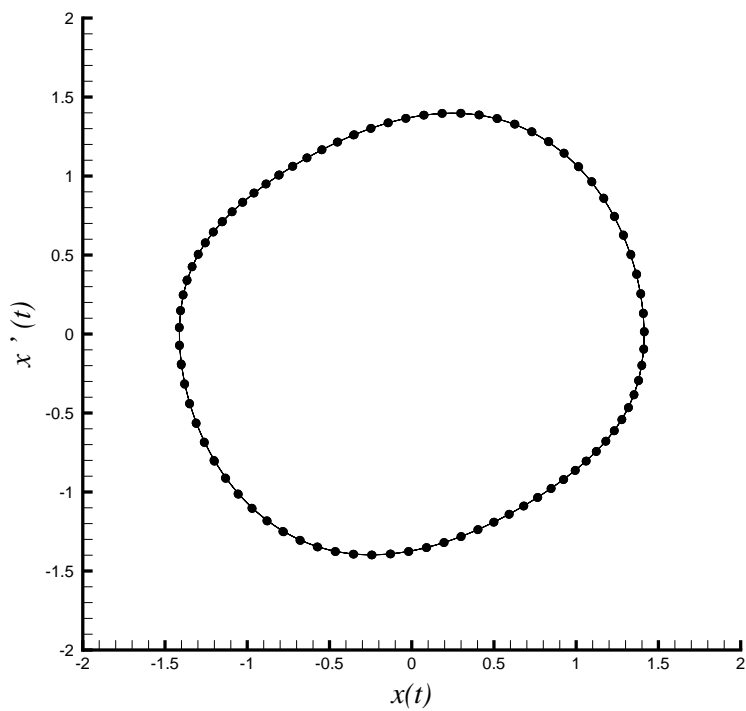


FIGURE 13.2

$x - \dot{x}$ plane projection of the limit cycle when $\epsilon = 1/5$ and $\mu = 3$. Solid line: fifth-order approximation by means of $\hbar_u = \hbar_v = -3/4$ and $H_u(\tau) = H_v(\tau) = 1$; symbols: numerical result.

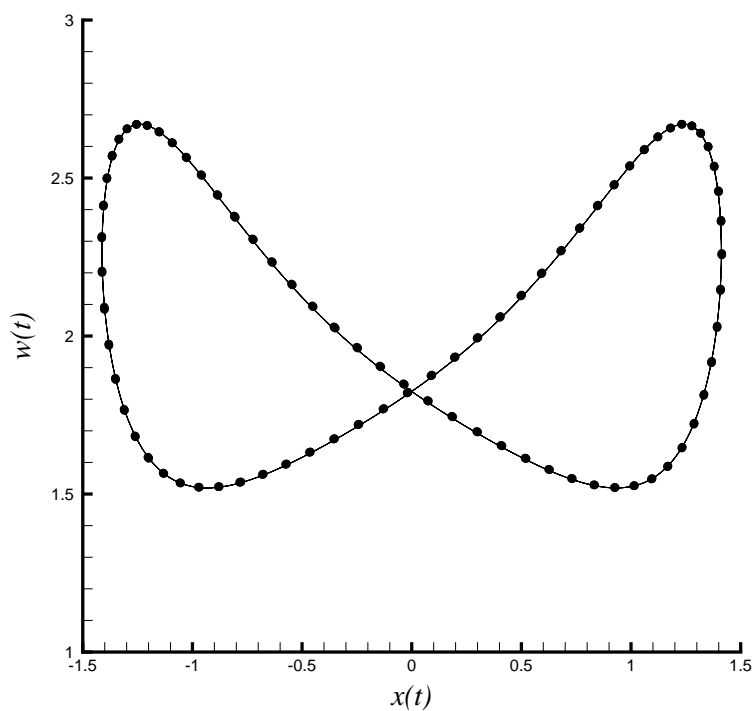


FIGURE 13.3

$x - w$ plane projection of the limit cycle when $\epsilon = 1/5$ and $\mu = 3$. Solid line: fifth-order approximation by means of $\hbar_u = \hbar_v = -3/4$ and $H_u(\tau) = H_v(\tau) = 1$; symbols: numerical result.

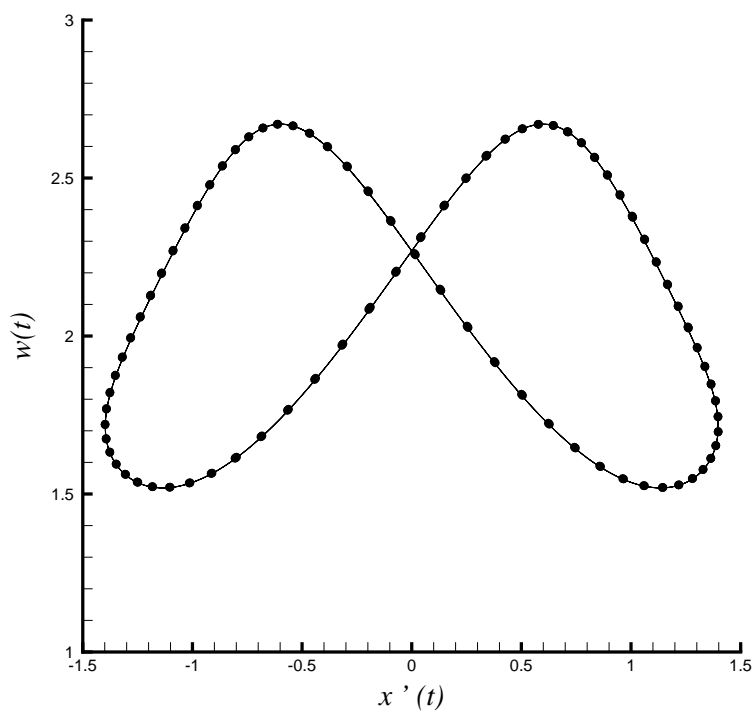


FIGURE 13.4

$\dot{x} - w$ plane projection of the limit cycle when $\epsilon = 1/5$ and $\mu = 3$. Solid line: fifth-order approximation by means of $\hbar_u = \hbar_v = -3/4$ and $H_u(\tau) = H_v(\tau) = 1$; symbols: numerical result.

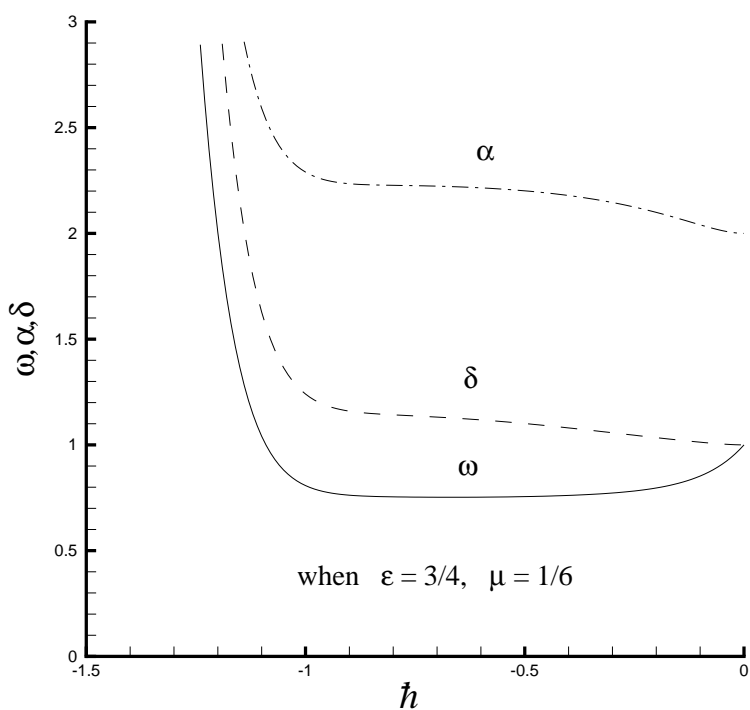


FIGURE 13.5

The \hbar -curves of ω , α , and δ at the 10th order of approximation when $\epsilon = 3/4$ and $\mu = 1/6$ by means of $H_u(\tau) = H_v(\tau) = 1$. Dashed line: δ ; dash-dotted line: α ; solid line: ω .

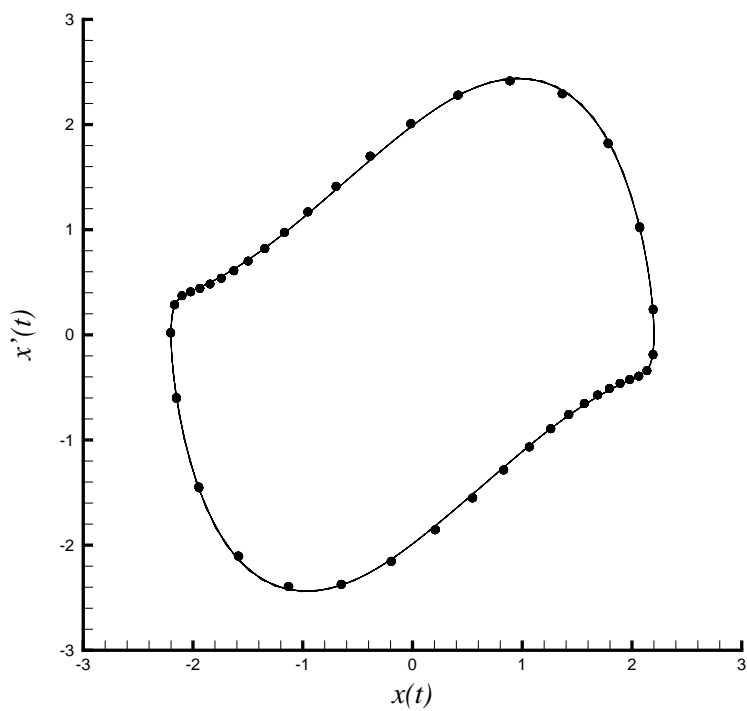


FIGURE 13.6

$x - \dot{x}$ plane projection of the limit cycle when $\epsilon = 3/4$ and $\mu = 1/6$. Solid line: 20th-order approximation by means of $\hbar_u = \hbar_v = -3/4$ and $H_u(\tau) = H_v(\tau) = 1$; Symbols: numerical result.

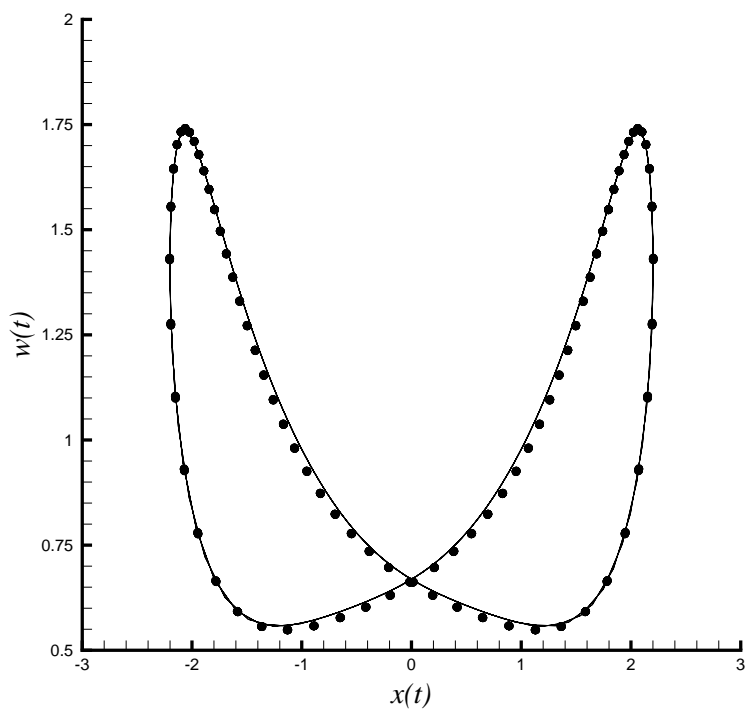


FIGURE 13.7

$x - w$ plane projection of the limit cycle when $\epsilon = 3/4$ and $\mu = 1/6$. Solid line: 20th-order approximation by means of $\hbar_u = \hbar_v = -3/4$ and $H_u(\tau) = H_v(\tau) = 1$; symbols: numerical result.

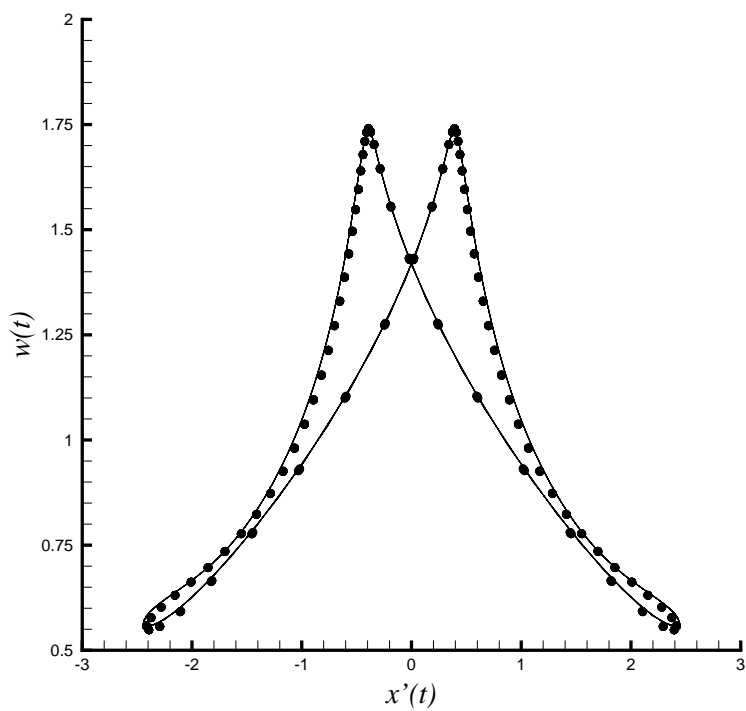


FIGURE 13.8

$\dot{x} - w$ plane projection of the limit cycle when $\epsilon = 3/4$ and $\mu = 1/6$. Solid line: 20th-order approximation by means of $\hbar_u = \hbar_v = -3/4$ and $H_u(\tau) = H_v(\tau) = 1$; symbols: numerical result.